Indian Institute of Management Calcutta

Working Paper Series

WPS No 856 /January 2021

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A Polyhedral Study of Generalized Assignment Problem with Demand Constraints

Prasenjit Mandal[1]

Abstract:

In this paper, I address a variant of traditional Generalized Assignment Problem (GAP) where multiple agents are assigned to jobs to satisfy the task requirements. I refer to this problem as a generalized assignment problem with demand constraints (GAPD). Like GAP, the GAPD is also a well-known (and NP-hard) combinatorial optimization problem. I present the well-known generic cover and (1, k)-configuration inequalities for a single job. Furthermore, I introduce several other classes of non-trivial valid inequalities involving multiple jobs.

Keywords: GAP with Demand Constraints, Valid Inequalities, Polyhedral Study

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1 Introduction

The traditional Generalized Assignment Problem (GAP) is a classical NP-hard discrete optimization problem. It consists of minimizing the assignment costs of a set of jobs to a set of machines while satisfying the capacity constraints. It is one of the most widely addressed problems in the integer programming and combinatorial optimization literature (Cattrysse and Van Wassenhove, 1992).

The purpose of this paper is to study a problem similar to the GAP where a set of agents with limited proficiency are assigned to a set of jobs to satisfy their demands. The demand constraints are typically the well-known knapsack inequalities in the form of greater-than-or-equal-to type constraints. Like the GAP, an agent can be assigned to one job only. I assume that the cost of assignment is proportional to the proficiency of the agent. I refer to this problem as generalized assignment problem with demand constraints (GAPD). Hence, it is a variant of the GAP.

GAPD has numerous real life applications and it may also appear as a sub-problem in several other problems. Although, I started with a problem that considers assignment of agents to jobs, problems with similar structures arise in many other real life scenarios. I provide a few such examples here. In a software development firm, managers often estimate the man-hour requirements for the ongoing projects and allocate a group of software professionals in form of teams to different projects to meet the requirements. Also, GAPD appears as a sub-problem to staff scheduling and rostering problem where a firm constructs work timetables for its staff to satisfy the demand for goods or services. The application areas of staff scheduling and rostering include health care systems, transportation services such as airlines and railways, emergency services such as police, ambulance and fire brigade, call centres, and other service firms like hotels, restaurants and retail stores (Ernst et al., 2004; Van den Bergh et al., 2013). Similar situations also arise in some other contexts such as load balancing in assembly lines. Motivated by the examples mentioned above, this paper introduces a family of valid inequalities for the GAPD.

The GAPD can be described as follows. Each agent has heterogeneous proficiencies in terms of man-hours to different jobs. Now, each job has its own man-hour requirements (demands). Hence, the agents are assigned to jobs to meet the man-hour requirements. Let, $\mathcal{M} := \{1, \dots, m\}$ be the set of agents and $\mathcal{N} := \{1, \dots, n\}$ be the set of jobs. If an agent ("she") $k \in \mathcal{M}$ is assigned to a job $j \in \mathcal{N}$, then she can contribute $a_{jk} \geq 0$ man-hours. Also, let $c_{jk} \geq 0$ be the wage paid to (cost of) agent $k \in \mathcal{M}$ for being assigned to job $j \in \mathcal{N}$. I denote $d_j \geq 0$ as the man-hour demand required to complete the job $j \in \mathcal{N}$. I assume that $m, n \geq 2$. The GAPD is formulated as the following integer program:

$$\min \sum_{j \in \mathcal{N}} \sum_{k \in \mathcal{M}} c_{jk} x_{jk}$$

$$s.t$$

$$\sum_{k \in \mathcal{M}} a_{jk} x_{jk} \geq d_j, j \in \mathcal{N} \tag{1}$$

$$\sum_{j \in \mathcal{N}} x_{jk} \leq 1, k \in \mathcal{M} \tag{2}$$

$$x_{jk} \in \{0, 1\}, j \in \mathcal{N}, k \in \mathcal{M} \tag{3}$$

In GAPD, the objective is to minimize the sum of wages paid to the agents. The first set of constraints (1) ensure the demand d_j (in terms of man-hour) of each job $j \in \mathcal{N}$ must be fully satisfied (demand constraints). Constraint set (2) enforces that every agent can be assigned to at maximum one job (SOS constraints). Although, a job can take multiple agents.

Let $\mathbf{d} = \{d_j\}_{j \in \mathcal{N}}$, $\mathbf{c} = \{c_{jk}\}_{j \in \mathcal{N}, k \in \mathcal{M}}$, $\mathbf{a} = \{a_{jk}\}_{j \in \mathcal{N}, k \in \mathcal{M}}$ denote the demand, cost and proficiency vectors corresponding to a data instance. Given an instance $\mathcal{N}, \mathcal{M}, \mathbf{d}, \mathbf{c}, \mathbf{a}$ of the GAPD, I define

$$X^{GAPD} = \Big\{ x \in \mathbb{R}^{|\mathcal{N}| \times |\mathcal{M}|} \Big| x \text{ satisfies } (1), (2), (3) \Big\}.$$

The purpose of this paper is to study the GAPD polytope, X^{GAPD}. Then the convex hull of the $0-1$ vertices of the GAPD polytope

$$P^{GAPD} = conv \Big\{ x \in \mathbb{R}^{\ |\mathcal{N}| \times |\mathcal{M}|} \Big| x \text{ satisfies } (1), (2), (3) \Big\}.$$

Assumption 1.

$$\sum_{k \in \mathcal{M}} \min_{j \in \mathcal{N}} a_{jk} \geq \sum_{j \in \mathcal{N}} d_j.$$

Assumption 1 states that $X_{GAPD} \neq \emptyset$, i.e., there exists at least a flexible solution.

The GAP is a widely addressed problem in the integer programming literature. Different types of heuristic and exact algorithms are presented in (Ross and Soland, 1975; Martello and Toth, 1981; Savelsbergh, 1997; Nauss, 2003). Cattrysse et al. Cattrysse et al. (1998) are the first to use a B&C algorithm to solve the GAP. Later, Avella et al. Avella et al. (2010) use an exact knapsack separation algorithm embedded into a B&C

scheme. A few papers also study the polyhedral structure of the GAP (Gottlieb and Rao, 1990b,a; De Farias and Nemhauser, 2001). These papers have introduced several types of valid inequalities that are valid to GAP polytope. Finally, this problem considers the knapsack inequalities are in the form of greater-than-or-equal-to types of inequalities. The polyhedral study on knapsack inequalities is well researched (Balas, 1975; Hammer et al., 1975; Balas and Zemel, 1978; Wolsey, 1990; Gokce and Wilhelm, 2015). Among them, Gokce et al. Gokce and Wilhelm (2015) presents valid inequalities for greater-than-or-equal-to types of knapsack inequalities, augmented by generalized upper bound (GUB) constraints.

Although the literature on GAP with capacity constraint is abundant, to the best of my knowledge none of them has addressed the GAP with a demand fulfilment constraint. The problem structure of GAP changes completely when the capacity constraints are replaced by demand constraints. Despite GAPD may appear as a relaxation to many integer programming problems, the literature to study the polyhedral properties of P_{GAPD} is quite limited. To the best of my knowledge, I am the first to address the GAPD problem and to study polyhedral structure of it. I derive a class of valid inequalities that has to describe underlying GAPD polytope P^{GAPD}. At first, I present the well-known generic cover and $(1, k)$-configuration inequalities for single job. Next, I derive several classes of non-trivial valid inequalities involving multiple jobs.

The remainder of the paper is organized as follows. In Section 2, I study the polyhedral structures of the proposed model and derive several classes of inequalities valid to this problem. In Section 3, I conclude the work.

2 Valid Inequalities for the GAPD

In this section, I study the polyhedral structure of GAPD. First, I prove that GAPD is a NP-Hard problem in a strong sense. Then I present two well-known classes of inequalities such as cover inequalities and $(1, k)$-configuration inequalities. After that I derive three different classes of inequalities which consider multiple jobs.

Proposition 1. *The GAPD is a NP-hard problem in the strong sense.*

Proof. The NP-hardness of GAPD can be easily proved by establishing that the well-known $3-$PARTITION problem is its special case (Garey and Johnson, 1979). Let the GAPD is restricted to special case by considering an instance with $c_{jk} = c_k$ for $k \in \mathcal{M}$; $a_{jk} = a_k$ for $k \in \mathcal{M}$. However, this restricted GAPD can be thought of a variant of a multiple knapsack problem (Martello, 1990). In fact, given these set of $2|\mathcal{M}|+|\mathcal{N}|$ positive integers $c_1, \ldots, c_{|\mathcal{M}|}$; $a_1, \ldots, a_{|\mathcal{M}|}$; $d_1, \ldots, d_{|\mathcal{N}|}$ and another positive integer α, this restricted GAPD investigates if there exists $|\mathcal{N}|$ disjoint subsets $S_1, S_2, \ldots, S_{|\mathcal{N}|}$ of $\mathcal{M}$ such that $\sum_{k \in S_j} a_k \geq d_j$ for $j \in \mathcal{N}$ and

$\sum_{j\in\mathcal{N}}\sum_{k\in S_j} c_k \leq \alpha$. Now, any data instance I of the 3−PARTITION problem can be pseudo-polynomially transformed, without loss of generality, into an equivalent instance $\hat{I}$ of the restricted GAPD (i.e., a case of multiple knapsack problem) by setting $d_j = B$ for $j \in \mathcal{N}$, $c_k = 1$ for $k \in \mathcal{M}$ and $\alpha = |\mathcal{M}|$ (Martello, 1990). As a 3−PARTITION problem is strongly NP-hard, the restricted GAPD is also NP-hard (Garey and Johnson, 1979). Hence, the GAPD as a generalization of the restricted GAPD must also be NP-hard. □

2.1 Individual Cover Inequalities

GAPD has a special structure. The problem consists of $|\mathcal{N}|$ number of greater-than-equal-to type of knapsack constraints. Let, $P_{KP}(j)$ denotes the knapsack polytope corresponding to job $j \in \mathcal{N}$. Then,

$$P_{KP}(j) = \Big\{ \sum_{k\in\mathcal{M}} a_{jk}x_{jk} \geq d_j | x_{jk} \in \{0,1\}, k \in \mathcal{M} \Big\}, \forall j \in \mathcal{N}.$$

The knapsack polytope $P_{KP}(j)$ is a relaxation of X_{GAPD}. Cover inequalities were introduced by Balas Balas (1975), Hammer et al. Hammer et al. (1975) and Balas et al. Balas and Zemel (1978) for a knapsack polytope. Later, Gottlieb and Rao Gottlieb and Rao (1990b) also derived the individual cover inequalities for GAP. Here I present similar inequalities for the $P_{KP}(j)$.

Definition 2.1. A set $C_j \subseteq \mathcal{M}, \forall j \in \mathcal{N}$ and $\bar{C}_j := \mathcal{M}\backslash C_j$. C_j is an individual cover for $j \in \mathcal{N}$ if

$$\sum_{k\in\bar{C}_j} a_{jk} < b_j.$$

If C_j is a cover for job $j \in \mathcal{N}$, then $\bar{C}_j$ is also defined as the anti-cover for j.

Definition 2.2. The set C_j is a minimal individual cover if

$$\sum_{k\in\bar{C}_j\cup\{\ell\}} a_{jk} \geq b_j$$

for all $\ell \in C_j$.

The definition of individual cover C_j provides a feasibility condition for P_{GAPD} polytope. At least one of the variables in cover C_j need to be equal to 1 to satisfy the feasible space. Next, the following proposition specifies the individual cover inequality.

Proposition 2. *The minimal individual cover inequality*

$$\sum_{k\in C_j} x_{jk} \geq 1 \tag{4}$$

is valid for P_{GAPD}.

Next, I introduce the extended individual cover to obtain stronger inequalities. For a minimal individual cover C_j, let $a_j^* := \max_{k \in C_j} a_{jk}$ and $E(C_j) = \{k \in \bar{C}_j | a_{jk} \geq a_j^*\}$. Then, the following set of inequalities are referred as extended individual cover inequalities:

$$\sum_{k \in C_j \cup E(C_j)} x_{jk} \geq 1 + |E(C_j)| \tag{5}$$

Similar to Gottlieb and Rao Gottlieb and Rao (1990b), I also derive the set individual $(1, k_j)$-configuration inequalities for each job.

Definition 2.3. For each $j \in \mathcal{N}$, a set $M_j^{'} \cup \{z\}$ is a $(1, k_j)$-configuration if $M_j^{'} \subset \mathcal{M}$, $|M_j^{'}| = m_j^{'}$ and $z \in \mathcal{M} \backslash M_j^{'}$ are such that

(i) $\sum_{k \in \mathcal{M} \backslash M_j^{'}} a_{jk} \geq d_j$,

(ii) $K_j \cup \{z\}$ is a minimal cover for each $K_j \subseteq M_j^{'}$ with $|K_j| = k_j$ where k_j is an integer satisfying $2 \leq k_j \leq m_j^{'}$ (i.e., elements in $\mathcal{M} \backslash \{K_j \cup \{z\}\}$ can't satisfy the demand d_j).

Proposition 3. *The individual* $(1, k_j)$*-configuration inequality*

$$(r_j - k_j + 1)x_{jz} + \sum_{k \in R_j} x_{jk} \geq (r_j - k_j + 1) \tag{6}$$

is valid for P_{GAPD}*, where* $R_j \subseteq M_j^{'}, |R_j| = r_j$ *satisfying* $k_j \leq r_j \leq m_j^{'}$.

If $k_j = m_j^{'}$, I observe that the individual $(1, k_j)$-configuration is a individual minimal cover.

2.2 Multiple Cover Inequalities

In this section, I restrict my attention to inequalities that consider multiple jobs. Next in Proposition 4, I present several classes of valid inequalities corresponding to a subset of jobs.

Proposition 4. *(a) For some job* $p \in N$*, let* $S \subset M$ *be a set of agents such that* S *is a cover, i.e.,* $\sum_{k \in \bar{S}} a_{pk} < d_p$*. Let,* $\underline{k}_p = \arg\min_{k \in \bar{S}} a_{pk}$*. There doesn't exist any agent* $v \in S$*, such that* $\sum_{k \in \bar{S} \backslash \{\underline{k}_p\}} a_{pk} + a_{pv} \geq d_p$*, i.e., substituting any agent from set* S *for the agent in* $\bar{S}$ *with minimum proficiency is not enough to satisfy the demand* d_p.

(b) For another job $l \in N, l \neq p$*, let* $\bar{T} \subset S$ *be a set of agents such that* $\bar{T} \bigcup \{s\}$ *is an anti-cover for all* $s \in \bar{S}$*, i.e.,* $\sum_{k \in \bar{T}} a_{lk} + a_{ls} < b_l$*. Equivalently, for all agent* $s \in \bar{S}$*, the set of agents* $T \backslash \{s\}$ *is denoted to be a cover for job* l*, where* $T = M \backslash \bar{T}$.

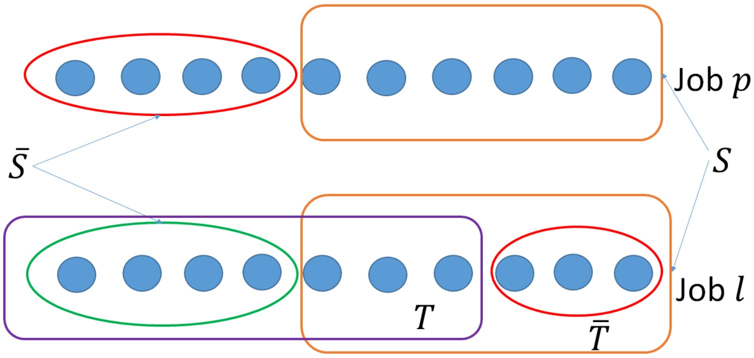

Figure 1: Multiple Cover Inequality

(c) Also, there doesn't exist any agent $t \in S \backslash \bar{T}$, such that the set of agents $\bar{T} \bigcup \{t\}$ satisfy the demand d_l, i.e. $\sum_{k \in \bar{T} \bigcup \{t\}} a_{lk} \ngeq d_l$.

Then the following inequality is valid for the P_{GAPD} polytope:

$$\sum_{k \in S} x_{pk} + \sum_{k \in T \backslash \bar{S}} x_{lk} \geq 3.$$

Proof. To prove the proposition, I consider three non-trivial cases.

Case 1: For a job p, let $x_{pk} = 1, k \in \bar{S}$, and for job l, let $x_{lk} = 1, k \in \bar{T}$. In that case, at least 1 additional resource is required to complete job p, whereas at least 2 additional resources are required to complete job l, i.e., $\sum_{k \in S} x_{pk} \geq 1$ and $\sum_{k \in T \backslash \bar{S}} x_{lk} \geq 2$.

Case 2: For a job p and for an agent $s \in \bar{S}$ let $x_{pk} = 1, k \in \bar{S} \backslash \{s\}$; whereas for job l, let $x_{lk} = 1, k \in \bar{T}$ and $x_{ls} = 1$. From Proposition 1(a), I know that $\sum_{k \in \bar{S} \backslash s} a_{pk} + a_{pv} < d_p$ for all agent $v \in S$; hence, at least 2 additional resources are required to complete job p, i.e., $\sum_{k \in S} x_{pk} \geq 2$. From Proposition 1(b), I know that $\bar{T} \bigcup \{s\}$ is an anti-cover and from Proposition 1(c), I know that $\sum_{k \in \bar{T} \bigcup \{t\}} a_{lk} < d_l$ for any agent $t \in S \backslash \bar{T}$. Hence, at least 1 additional resource is required to complete job l, i.e., $\sum_{k \in T \backslash \bar{S}} x_{lk} \geq 1$.

Case 3: For a job p and for any two agents $s_1, s_2 \in \bar{S}$ let $x_{pk} = 1, k \in \bar{S} \backslash \{s_1, s_2\}$ and for an agent $t \in \bar{T}$ (i.e., $t \in S$ as $S \supset \bar{T}$) , $x_{pt} = 1$; whereas for job l, let $x_{lk} = 1, k \in \bar{T} \backslash \{t\}$ and $x_{l,s_1} = 1, x_{l,s_2} = 1$. In that case, from Proposition 1(a), it can be easily shown that at least 1 additional resource is required to complete the job p, i.e., $\sum_{k \in S} x_{pk} \geq 2$. From Proposition 1(b) and Proposition 1(c), it can also be easily shown that at least one additional resource is required to complete job l, i.e., $\sum_{k \in T \backslash \bar{S}} x_{lk} \geq 1$.

For all these three non-trivial cases presented above, I need exactly 3 agents to complete both the jobs p and l. For all other trivial cases, it can easily shown that the minimum number of agents required to both the jobs p and l are at least 3. Hence, the inequality $\sum_{k \in S} x_{pk} + \sum_{k \in T \backslash \bar{S}} x_{lk} \geq 3$ is a valid one. It completes the proof of the proposition. □

Example 1. *Let us consider an example with 2 jobs and 5 agents. The constraints to the problem is given by:*

$$4x_{11} + 3x_{12} + 5x_{13} + 4x_{14} + 3x_{15} \geq 7$$

$$3x_{11} + 4x_{12} + 5x_{13} + 2x_{14} + 3x_{15} \geq 8$$

Let, $p = 1$ and $\bar{S} = \{1\}$, then cover set $S = \{2, 3, 4, 5\}$. Set $\{1\}$ be anti-cover for job 1 as $4 < 7$ (doesn't satisfy demand). The inequality $x_{12} + x_{13} + x_{14} + x_{15} \geq 1$ is a cover inequality for job 1.

Let, $l = 2$ and $\bar{T} = \{4\}$, then $T = \{1, 2, 3, 5\}$. For all $s \in \bar{S} = \{1\}$, $\bar{T} \cup \{s\}$ is an anti-cover. The cover for job 2 is the set $T \backslash \{s\}, \forall s \in \bar{S}$. Then, the inequality $x_{12} + x_{13} + x_{15} \geq 1$ is cover inequality for job 2. A cover inequality considering multiple jobs is given by:

$$\sum_{k=2}^{5} x_{1k} + \sum_{k \in \{2,3,5\}} x_{2k} \geq 3.$$

The set of all feasible integer points are given below. The inequality above satisfies all the feasible integer points. At the same time, please check that, for $l = 2$, if $T = \{1, 2, 3, 4\}$, then condition 4(a) and 4(b) are satisfied but condition 4(c) is not satisfied. Because for an agent $t = \{3\}$ and $\bar{T} = \{5\}$ (as $T = \{1, 2, 3, 4\}$), $a_{23} + a_{25} = 5 + 3 = 8$ exactly satisfies the demand $d_2 = 8$. Hence, $\sum_{k=2}^{5} x_{1k} + \sum_{k \in \{2,3,4\}} x_{2k} \geq 3$ can not be a multiple cover. From the set of feasible points, the first point $\begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$ presented in figure 1, doesn't satisfy this inequality. Hence, it is not a valid one.

Next in Corollary 1, I generalize the multiple cover inequality presented in proposition 4.

Corollary 1. *(a) Let, $W \subset N$ be a set of jobs. For some job $p \in N$, let $S \subset M$ be a set of agents such that S is a cover, i.e., $\sum_{k \in \bar{S}} a_{pk} < b_p$. There doesn't exist any agent $s \in S$, such that $\sum_{k \in \bar{S} \backslash \{\underline{k}_p\}} a_{pk} + a_{ps} \geq d_p$, i.e., substituting any agent from set S for the agent in $\bar{S}$ with minimum proficiency is not enough to satisfy the demand d_p.*

(b) For each job $j \in W \backslash \{p\}$, there exists a set of agents $\bar{T}_j \subset S$, $\bar{T}_{j_1} \bigcap \bar{T}_{j_2} = \emptyset, j_1 \neq j_2, j_1, j_2 \in W \backslash \{p\}$, such that $\bar{T}_j \bigcup \{s\}$ is an anti-cover for all agent $s \in \bar{S}$, i.e., $\sum_{k \in \bar{T}_j} a_{jk} + a_{js} < b_j$. Also, there doesn't exist any agent $t_j \in S \backslash \bar{T}_j$, such that the set of agents $\bar{T}_j \bigcup \{t_j\}$ satisfy the demand d_j, i.e. $\sum_{k \in \bar{T} \bigcup \{t_j\}} a_{jk} \ngeq d_j$.

Then the following inequality is valid for the P_{GAPD} polytope:

$$\sum_{k\in S} x_{pk} + \sum_{j\in W\backslash\{p\}} \sum_{k\in T_j\backslash \bar{S}} x_{jk} \geq |W| + 1.$$

The proof for Corollary 1 is essentially the same as for Proposition 4. Next in Proposition 5, I present another variant of multiple cover inequality.

Proposition 5. *(a) For some job $p \in W$, $W \subset N$ and a set of jobs $\bar{C} \subset M$, $|\bar{C}| = c$ is such that $\bar{C}$ is an anti-cover ($C = M\backslash\bar{C}$ is a cover), i.e., $\sum_{k\in\bar{C}} a_{pk} < d_p$. Let, $\underline{k}_p = \arg\min_{k\in\bar{C}} a_{pk}$. There doesn't exist any agent $s \in C$, such that $\sum_{k\in\bar{C}\backslash\{\underline{k}_p\}} a_{pk} + a_{ps} \geq d_p$, i.e., substituting any agent from set C for the agent in $\bar{C}$ with minimum proficiency is not enough to satisfy the demand d_p.*

(b) for each job $j \in W\backslash\{p\}$, there exists a set of agents $\bar{T}_j \subset C$ and

$$\bar{C}_j = \left\{k \in \bar{C} | \bar{T}_j \bigcup \{k\} \text{ is an anticover for job } j\right\},$$

with $|\bar{C}_j| = c_j > 0$.

(c) Also, there doesn't exist any agent $t_j \in C\backslash\bar{T}_j$, such that the set of agents $\bar{T}_j \bigcup \{t_j\}$ satisfy the demand d_j, i.e. $\sum_{k\in\bar{T}\bigcup\{t_j\}} a_{jk} \ngeq d_j$.

Then

$$\sum_{k\in C} x_{pk} + \sum_{j\in W\backslash\{p\}} \sum_{k\in T_j\backslash \bar{C}_j} x_{jk} \geq |W\backslash\{p\}| + \left\lceil \frac{(c_m - 1)(\mu_p - c) + 1}{c_m} \right\rceil,$$

is a valid inequality for the P_{GAPD} polytope, where $c_m = \max\{c_j | j \in W\backslash\{p\}\}$.

Proof. I know from 5(a) ,

$$\sum_{k\in C} x_{pk} \geq 1$$

From condition 5(b), I can write

$$\sum_{k\in T_j} x_{jk} - x_{ju} \geq 1, \forall u \in \bar{C}_j$$

From SOS constraint,

$$1 \geq \sum_{j\in W} x_{jk}, \forall k \in \bar{C}.$$

Now,

$$\begin{aligned}
&\frac{1}{c_m}\sum_{k\in C} x_{pj} + \sum_{j\in W\backslash\{p\}} \left\{\frac{1}{c_j}\sum_{u\in\bar{C}_j}\left(\sum_{k\in T_j} x_{jk} - x_{ju}\right)\right\} + \frac{c_m-1}{c_m}c \\
&\geq \frac{1}{c_m} + \sum_{j\in W\backslash\{p\}} \frac{c_j}{c_j} + \frac{c_m-1}{c_m}\sum_{k\in\bar{C}}\sum_{j\in W} x_{jk} \\
&= \frac{1}{c_m} + |W\backslash\{p\}| + \frac{c_m-1}{c_m}\sum_{k\in\bar{C}}\sum_{j\in W} x_{jk} \\
&= \frac{1}{c_m} + |W\backslash\{p\}| + \frac{c_m-1}{c_m}\sum_{k\in M\backslash C} x_{pk} + \frac{c_m-1}{c_m}\sum_{j\in W\backslash\{p\}}\sum_{k\in\bar{C}} x_{jk}
\end{aligned}$$

Re-arranging all the terms of the inequality above, I get

$$\begin{aligned}
&\frac{1}{c_m}\sum_{k\in C} x_{pj} + \frac{c_m-1}{c_m}\sum_{k\in C} x_{pk} + \sum_{j\in W\backslash\{p\}} \left\{\frac{1}{c_j}\sum_{u\in\bar{C}_j}\left(\sum_{k\in T_j} x_{jk} - x_{ju}\right)\right\} - \frac{c_m-1}{c_m}\sum_{j\in W\backslash\{p\}}\sum_{k\in\bar{C}} x_{jk} \\
\geq& \frac{1}{c_m} - \frac{c_m-1}{c_m}c + |W\backslash\{p\}| + \frac{c_m-1}{c_m}\sum_{k\in M} x_{pk} \\
\geq& |W\backslash\{p\}| + \frac{1}{c_m} - \frac{c_m-1}{c_m}c + \frac{c_m-1}{c_m}\mu_p \\
=& |W\backslash\{p\}| + \frac{(c_m-1)(\mu_p - c) + 1}{c_m}
\end{aligned}$$

where $\mu_p = \left\lceil \frac{d_p}{\bar{a}_p} \right\rceil$ with $\bar{a}_p = \max_{k\in\mathcal{M}} a_{pk}$ (μ_p is the minimum number of agents required to complete the job p). The second inequality holds true because $\sum_{k\in M} x_{pk} \geq \mu_p$.

Next, the LHS of the inequality above can be expressed as

$$
\begin{aligned}
&\frac{1}{c_m}\sum_{k\in C} x_{pj} + \frac{c_m-1}{c_m}\sum_{k\in C} x_{pk} + \sum_{j\in W\setminus\{p\}} \left\{ \frac{1}{c_j}\sum_{u\in \bar{C}_j}\left(\sum_{k\in T_j} x_{jk} - x_{ju}\right)\right\} - \frac{c_m-1}{c_m}\sum_{j\in W\setminus\{p\}}\sum_{k\in \bar{C}} x_{jk} \\
=&\sum_{k\in C} x_{pj} + \sum_{j\in W\setminus\{p\}} \left\{ \frac{c_j}{c_j}\sum_{k\in T_j} x_{jk} - \frac{1}{c_j}\sum_{u\in \bar{C}_j} x_{ju} - \frac{c_m-1}{c_m}\sum_{k\in \bar{C}} x_{jk} \right\} \\
=&\sum_{k\in C} x_{pj} + \sum_{j\in W\setminus\{p\}} \left\{ \sum_{k\in T_j} x_{jk} + \sum_{k\in \bar{C}_j}\left(-\frac{1}{c_j} - \frac{c_m-1}{c_m}\right) x_{jk} - \frac{c_m-1}{c_m}\sum_{k\in \bar{C}\setminus\bar{C}_j} x_{jk} \right\} \\
\le&\sum_{k\in C} x_{pj} + \sum_{j\in W\setminus\{p\}} \left\{ \sum_{k\in T_j} x_{jk} - \sum_{k\in \bar{C}_j}\frac{c_m + c_j(c_m-1)}{c_j c_m} x_{jk} \right\} \\
\le&\sum_{k\in C} x_{pj} + \sum_{j\in W\setminus\{p\}} \left\{ \sum_{k\in T_j} x_{jk} - \sum_{k\in \bar{C}_j} x_{jk} \right\},
\end{aligned}
$$

as $1 \le \frac{c_m + c_j(c_m-1)}{c_j c_m} < 2$.

Therefore, $\sum_{k\in C} x_{pj} + \sum_{j\in W\setminus\{p\}}\sum_{k\in T_j\setminus\bar{C}_j} x_{jk} \ge |W\setminus\{p\}| + \left\lceil \frac{(c_m-1)(\mu_p - c)+1}{c_m} \right\rceil$. □

2.3 Flow Cover Inequalities

Padberg et al. (1985) is the first to introduce flow cover inequalities for a network problems with fixed charges on the arcs. I incorporate similar set of inequalities here, although they are very weak, being dominated by other classes of valid inequalities. However, flow cover inequalities can be up-lifted to make them stronger.

Definition 2.4. A set of agents $K \subseteq \mathcal{M}$ is a flow cover for job $p \in \mathcal{N}$ if

(i) $\sum_{k\in\bar{K}} a_{pk} < d_p$, where $\bar{K} = \mathcal{M}\setminus K$,

(ii) $\sum_{k\in\left\{\bar{K}\cap\{t\}\right\}} a_{pk} \ge d_p$ for some $t \in K$.

Proposition 6. *Let, $\bar{K} \subset \mathcal{M}$, where $\bar{K} = \mathcal{M}\setminus K$ and $p \in \mathcal{N}$ such that $\sum_{k\in\bar{K}} a_{pk} < d_p$ and $\sum_{k\in\bar{K}\cup\{\ell\}} a_{pk} \ge d_p$ for some $\ell \in K$. Then*

$$
\sum_{k\in K} a_{pk}x_{pk} + \sum_{k\in K}\min\{\lambda, a_{pk}\}\left(1 - \sum_{j\in\mathcal{N}\setminus\{p\}} x_{jk}\right) \ge \lambda,
$$

where $\lambda = d_p - \sum_{k\in\bar{K}} a_{pk}$ is valid for the P_{GAPD} polytope.

Proof. I use the induction method and directly prove the proposition.

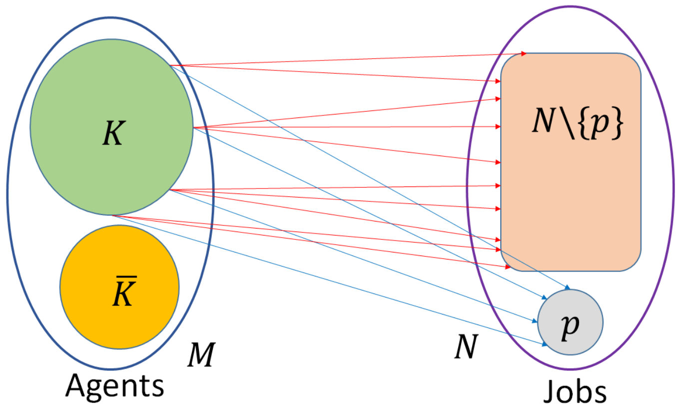

Figure 2: Multiple Cover Inequality

By definition of the flow cover, if the agents in the set $\bar{K} \subset \mathcal{M}$ is already assigned to job $p \in \mathcal{N}$, then the residual demands that are required to fulfil are,

$$\sum_{k \in K} a_{pk} x_{pk} \geq \lambda.$$

If any agent $\ell_1 \in K$ is assigned to job p by keeping all the agents in $K \backslash \{\ell_1\}$ left unassigned, then the minimum flow required to fulfil the demand d_p are

$$\sum_{k \in K \backslash \{\ell_1\}} a_{pk} x_{pk} = \max \{\lambda - a_{p\ell_1}, 0\} = \lambda - \min \{\lambda, a_{p\ell_1}\}.$$

I extend it further by induction. If any two agents $\ell_1, \ell_2 \in K$ are assigned to job p by keeping all the agents in $K \backslash \{\ell_1, \ell_2\}$ left unassigned, then the minimum flow required to fulfil the demand d_p are

$$\sum_{k \in K \backslash \{\ell_1, \ell_2\}} a_{pk} x_{pk} = \max \{\lambda - a_{p\ell_1} - a_{p\ell_2}, 0\} = \lambda - \min \{\lambda, a_{p\ell_1} + a_{p\ell_2}\} \geq \lambda - \sum_{k \in \{\ell_1, \ell_2\}} \min \{\lambda, a_{pk}\},$$

as $\sum_{k \in \{\ell_1, \ell_2\}} \min \{\lambda, a_{pk}\} \geq \min \{\lambda, a_{p\ell_1} + a_{p\ell_2}\}$.

Extending it further for any subset $\mathcal{L} \subset K$ where all the agents in $\mathcal{L}$ are assigned to job p by keeping all the agents in $K \backslash \mathcal{L}$ left unassigned, then the minimum flow required to fulfil the demand d_p are

$$= \sum_{k \in K \backslash \mathcal{L}} a_{pk} x_{pk} = \max \left\{ \lambda - \sum_{k \in \mathcal{L}} a_{pk}, 0 \right\} = \lambda - \min \left\{ \lambda, \sum_{k \in \mathcal{L}} a_{pk} \right\} \geq \lambda - \sum_{k \in \mathcal{L}} \min \{\lambda, a_{pk}\},$$

as $\sum_{k \in \mathcal{L}} \min \{\lambda, a_{pk}\} \geq \min \left\{\lambda, \sum_{k \in \mathcal{L}} a_{pk}\right\}$. Generalizing the inequality above, I get

$$
\begin{aligned}
&\sum_{k\in K} a_{pk}x_{pk} \geq \lambda - \sum_{k\in K} \min\{\lambda, a_{pk}\}\left(1 - \sum_{j\in\mathcal{N}\setminus\{p\}} x_{jk}\right)\\
\Rightarrow &\sum_{k\in K} a_{pk}x_{pk} + \sum_{k\in K} \min\{\lambda, a_{pk}\}\left(1 - \sum_{j\in\mathcal{N}\setminus\{p\}} x_{jk}\right) \geq \lambda,
\end{aligned}
$$

so the inequality is valid. □

3 Conclusion and Recommendations for Future Research

This paper establishes several valid inequalities to solve the GAPD effectively. Thus, I study the polyhedral properties of the convex hull of the GAPD which comprises of a set of greater-than-equal-to types of knapsack inequalities (each knapsack corresponds to a job) with SOS constraints. The GAPD appears as a relaxation of several optimization problems. I introduce several families of valid inequalities for the GAPD: the multiple cover and flow cover inequalities. As a future research direction, I would like to develop appropriate combinatorial separation algorithms for these inequalities. Future research could also contribute by seeking a disaggregated formulation of the GAPD.

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