Indian Institute of Management Calcutta

Working Paper Series

WPS No. 885/ March 2023

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Characterization of *maxmed* mechanisms for multiple objects

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Abstract

This paper presents an extension of maxmed mechanisms introduced by Sprumont [23] to the multiple homogeneous objects setting. To address the complexities of the multiple object setting, we consider special families of mechanisms which contain a mechanism for each possible number of available objects. We interpret these families as ex-ante sale procedures which specify different mechanisms to allocate different quantities of objects. We identify, and completely characterize the *maxmed families* which use the (extended) maxmed mechanisms to allocate any number of available objects while using the same non-negative reserve price. The maxmed families turn out to be the only families that are Pareto optimal among well behaved families comprising mechanisms that satisfy a set of desirable axioms including (the ones used by Sprumont [23]): anonymity, strategyproofness, no-envy, feasibility and individual rationality.

JEL classification: C72; C78; D71; D63

Keywords: Maxmed mechanism, strategyproof mechanism, multiple object allocation

1 Introduction

Sprumont [23] studies the problem of identifying Pareto optimal mechanisms for the single object allotment problem with money. He considers decision inefficient mechanisms outside the class of VCG mechanisms, and attempts to identify the Pareto frontier of the class of feasible strategyproof mechanism. To keep the analysis tractable, he focuses on anonymous and non-envious strategyproof mechanisms, and introduces a new class of "*maxmed*" mechanisms.[1] He shows that these are the only Pareto optimal mechanisms

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[1] Both anonymity and no-envy are well know fairness notions. The former requires mechanisms to ignore agent identities while deciding allocations for a reported profile. The latter requires that for every reported valuation, no agent strictly prefers the allocation bundle of another agent than her own allocation.

in the class of anonymous, non-envious, feasible, individually rational, and strategyproof mechanisms. In the present paper, we extend his results to the multiple identical object setting by providing an extension of the maxmed mechanism functional form to the multiple object setting.

Allowing multiple objects complicates the analysis substantially as any one agent getting an object no longer implies that every other agents gets no object. Furthermore, with multiple objects, there is a proliferation of allocation choices available to the planner at any reported valuation profile, because now she can choose to *not* allot all available objects.[2] Hence, to obtain a characterization on the lines of Sprumont [23] with multiple objects, it becomes necessary to use a restriction on the behaviour of mechanisms as the number of objects being allocated changes. To accommodate such a restriction, we study the problem in terms of "families" of mechanisms which contain a specific mechanism for each possible number of units k that may be available for allocation. Thus, in our setting, a social planner must choose a family of mechanisms to execute the allocation exercise prior to the realization of the actual number of objects to be allocated.[3]

This conceptualization of families of mechanisms allows us to motivate a *regularity* condition, also used in Basu and Mukherjee [4], which requires that set of valuation profiles where no objects are allocated - to not shrink when the number of units available for allocation increases.[4] We analyze the class of regular families, which contain continuous, anonymous, feasible, individually rational, strategyproof mechanisms that satisfy non-bossiness in decision.[5] In particular, we identify families F which are Pareto optimal among all families that comprise of anonymous, continuous, feasible, individually rational, non-bossy in decision, non-envious, and strategyproof mechanisms. We show that these Pareto optimal families are same as the ones that use maxmed mechanisms to allocate different supplies of available objects while using the *same* non-negative reserve price. We call these the 'maxmed families' of mechanisms, and thus, present a complete characterization of the maxmed families.

Anonymity requires that the welfare obtained from bidding in a mechanism not depend on agent identities. Non-bossiness in decision requires that no agent be able to influence allotment decision of another agent without changing her own allotment decision.[6] Feasibility requires that the mechanism not entail wastage (so that sum of transfers

[2] So if three objects are available, then she can choose to allocate any $k \in \{0, 1, 2, 3\}$ objects.

[3] Such a setting is observed in many real life situations. For example, an auctioneer (government ministry) decides the modalities of auctions - before the exact number of objects (tenders) available for sale gets decided. Sometimes the same predetermined auction format gets used in multiple consecutive years with different numbers of available objects. Note that our notion of families of mechanisms allows a planner to plan use of different mechanisms for different quantities of available licenses or antiquities.

[4] To be exact, we require this set to remain unchanged.

[5] We use the same notion of continuity as in Basu and Mukherjee [4].

[6] Similar notions of non-bossiness have been used by Satterthwaite and Sonnenschein [21], Svensson [24], Goswami, Mitra and Sen [9] etc. This version has also been used by Basu and Mukherjee [4] and Mishra and Quadir [14].

is never positive); while individual rationality implies that agents are not penalized for participating in the mechanism (so that utility obtained by bidding is never negative). Continuity of a mechanism ensures that mechanism outcomes do not change arbitrarily for small changes in bid values, and strategyproofness ensures truth telling is a weakly dominant strategy for all agents in the ensuing message game.

2 Literature Review

As mentioned above, our work is an extension of Sprumont [23] to the multiple identical object setting. Apart from Sprumont [23], our work also relates to the papers on optimal strategyproof mechanisms to allocate multiple objects. Some such notable papers are Apt, Conitzer, Guo and Markakis [1], Athanasiou [3], Guo and Conitzer [11], Guo and Conitzer [12], Moulin [16], Moulin [17], Ohseto [20]. While these papers differ in terms of the class of mechanisms considered and the optimality notion used, all of them assume allotment decision efficiency and hence, limit their study to the class of VCG mechanisms (Vickrey [27], Clarke [5], Groves [10]).

Some papers which consider the problem of welfare maximization while allowing for deterministic mechanisms without allotment efficiency are: de Clippel, Naroditskiy and Greenwald [6], Drexl and Kleiner [7], Shao and Zhu [22]. Drexl and Kleiner [7] focuses on strategyproof, individually rational and feasible mechanisms in a two agent setting, and uses a prior distribution to specify the expected aggregate utility maximizing mechanism. Shao and Zhu [22] obtains similar results as Drexl and Kleiner [7] without the use of individual rationality, but with a more restrictive distribution of types. de Clippel, Naroditskiy and Greenwald [6] considers a multiple homogeneous setting like ours, and presents a feasible, anonymous, strategyproof, individually rational and decision inefficient mechanism that distributes at least eighty percent of the social welfare generated as number of agents goes to infinity.

However, the papers that are closest to ours are Sprumont [23], and Athanasiou [3] - both of which use the single object setting. Athanasiou [3] presents necessary conditions that a strategyproof and anonymous mechanism must satisfy to be Pareto optimal among the class of all such mechanisms.[7] He identifies maxmed mechanisms as a class of Pareto optimal mechanisms among all strategyproof and anonymous mechanisms when there are two agents. In contrast, we consider a mulitple of object setting like de Clippel, Naroditskiy and Greenwald [6], and provide a complete characterization of the maxmed families (that use maxmed mechanisms to allocate any possible quantity of available objects). We show that these are the only Pareto optimal families among all regular families, which use anonymous, continuous, feasible, individually rational, non-bossy, non-envious, and strat-

[7] Athanasiou [3] also shows that when coupled with individual rationality, these necessary conditions become sufficient.

egyproof mechanisms. Thus, our paper specifies the exact functional form of maxmed mechanisms when extended to the multiple homogeneous object allocation setting.

3 Model

We consider a situation where m homogeneous indivisible objects are to be allotted to agents in $N = \{1, 2, \ldots, n\}$ with unit demand with $m < n$. Each agent $i \in N$ has an independent private valuation $v_i \in \mathbb{R}_+$. For any $i \in N$, a generic allocation of i is denoted by (d_i, t) where d_i represents the object allotment decision taking values in $\{0, 1\}$ with $d_i = 1$ if and only if i gets an object, and t represents an amount of money. We assume that agents have quasilinear preferences over object and money, that is, utility to i from the allocation (d_i, t) is $d_i v_i + t$.

A mechanism is a tuple of functions (d^m, τ^m) such that at any reported profile of valuations $v \in \mathbb{R}^N_+$, each agent i is allocated a monetary transfer $\tau^m_i(v) \in \mathbb{R}$ and a decision $d^m_i(v) \in \{0, 1\}$. For any reported valuation profile $v \in \mathbb{R}^N_+$, define $W^m(v) := \{i \in N | d^m_i(v) = 1\}$ to be the set of agents that are allocated an object. Note that at any reported profile of valuations $v \in \mathbb{R}^N_+$, $|W^m(v)| \leq m$, that is, all objects need not get allocated at all reported profiles. Therefore, the utility to any agent i with a true valuation of v_i at any reported profile $v' \in \mathbb{R}^N_+$, from the mechanism (d^m, τ^m) is given by $u((d^m_i(v'), \tau^m_i(v')); v_i) = v_i d^m_i(v') + \tau^m_i(v')$. For any $m \geq 2$, let $\mathcal{A}^m$ be the set of all possible mechanisms to allocate m objects.

As mentioned earlier, in this paper, we focus on *family* of mechanisms that describe procedures to allocate any number of homogeneous objects. Such a family is a list of mechanisms specifying one mechanism for each possible quantity of homogeneous object supply. Thus, a family of mechanisms represents an *ex-ante* sale procedure, that is chosen and fixed prior to the realization of the number of objects to be available for allotment.[8] Let $\bar{\mathcal{A}}$ be the set of all such families of mechanisms, that is, $\bar{\mathcal{A}} := \Pi_{m \in \mathbb{N}} \mathcal{A}^m$. Also, let $F = \{F^1, F^2, \ldots\}$ denote a generic family of mechanisms in $\bar{\mathcal{A}}$, with the interpretation that the mechanism F^m is to be used to allot objects when the number of available objects turns out to be m. In this paper, we focus on well behaved families of mechanisms which satisfy some degree of monotonicity as defined below.

Definition 1. A family of mechanisms $F \in \bar{\mathcal{A}}$ is said to be *regular* if for all $m > m' \in \mathbb{N}$,

$$\{v \in \mathbb{R}^N_+ : \textit{ no objects are allocated at } v \textit{ by } F^m\} = \{v \in \mathbb{R}^N_+ : \textit{ no objects are allocated at } v \textit{ by } F^{m'}\}.$$

[8] The implicit assumption here is that at any date, the number of potential buyers n will be greater than the number of available objects $m \geq 2$. We could easily avoid making this assumption by redefining a family of mechanisms as $F = (F^1, F^2, \ldots, F^{n-1})$. In that case, all results and arguments of our paper would continue to hold with some notational modifications. We do not do so for the sake of notational simplicity, and consistency with Basu and Mukherjee [4], on whose results we base ours.

Let $\bar{\mathcal{A}}_r$ denote the set of regular families of mechanisms.

Thus, a regular family of mechanisms displays a monotonicity property such that the set of profiles where no objects are allocated does not expand as the number of available objects increases. This restriction rules out peculiar ex-ante sale procedures where abundance of objects leads to scarcity in allocations.

Finally, we use the following notations. Let $\forall \{i,j\} \subseteq N$, $\forall v \in \mathbb{R}_+^N$, $v_{-i} := (v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$ and $v_{-i-j} := (v_{-i})_{-j}$. Further, for all $r = 1, 2 \ldots, n$, define $v(r)$ to be the rth ranked valuation in a non-increasing arrangement of coordinates of any $v \in \mathbb{R}_+^N$. In case of ties while arranging the coordinates in such manner, without loss of generality, we use the tie-breaking rule $1 \succ \ldots \succ n$.[9] Further, define for any $x \geq 0$, $\bar{x}^t := (x, x, \ldots, x) \in \mathbb{R}_+^t$ for all $t = 1, 2, \ldots, n$. Therefore, $\bar{x}^n = (x, x, \ldots, x) \in \mathbb{R}_+^n$ and $\bar{x}^{n-1} = (x, x, \ldots, x) \in \mathbb{R}_+^{n-1}$. Finally, for any three real numbers x, y, z, define $med\{x, y, z\}$ to be the median of these three numbers.

3.1 Axioms and other definitions

We now present the formal definitions of the axioms - ethical, strategic and technical - that we will use in this paper. We begin by first defining a technical axiom of continuity below, which requires that: whenever the allocation decision of an agent i is not preserved in limit, the transfer assigned to i at the limit profile is such that she is indifferent between getting and not getting the object.

Definition 2. A mechanism (d^m, τ^m) is said to be *continuous* if for any $\zeta \in \{0, 1\}$, any $i \in N$ and any sequence of profiles $\{v^k\}$ that converges to $\tilde{v}$, whenever $d_i(v^k) = \zeta$ for all k,

$$d_i(\tilde{v}) \neq \zeta \implies u((1, \tau_i(\tilde{v}; d_i = 1); \tilde{v}_i) = u((0, \tau_i(\tilde{v}; d_i = 0); \tilde{v}_i).$$

Let $\mathcal{A}_c^m$ denote the set of continuous mechanisms to allot a supply of m objects, and $\bar{\mathcal{A}}_c := \Pi_{m \in \mathbb{N}} \mathcal{A}_c^m$ denote all possible families that comprise continuous mechanisms.

Note that the transfer functions of strategyproof mechanisms may depend on any valuation profile v, in an indirect manner through the allotment decision's dependence on v. For example, for Vickrey auction with or without reserve price, the transfer at any profile depends not only on the profile but also on the allocation decision at that profile. In fact, a Vickrey auction is a continuous mechanism because at all profiles where the top bidders (at least two in number) bid the same value, everyone gets zero utility irrespective of which bidders win the object.[10]

[9] For any $i \neq j$, $i \succ j$ means that the tie is broken in favour of agent i. That is, for any v, if $v_3 = v_7 > v_i$ for all $i \neq 3, 7$ and $3 \succ 7$, then $v(1) = v_3$.

[10] To see the kind of mechanisms ruled out by the restriction of continuity, consider a setting where $m = 2$ and $n = 3$. Fix a mechanism such that at all bid profiles, it allocates both objects to the first

In the second definition below, we state the extension of the class of *maxmed* mechanisms, which were introduced by Sprumont [23] for a single object setting, to the present multiple identical object setting.

Definition 3. Any mechanism $(d^{m,r}, \tau^{m,r}) \in \mathcal{A}^m$ is said to be a *maxmed with reserve price* $r \geq 0$ if for all $i \in N$ and all $v \in \mathbb{R}_+^N$,

- $v_i < \max\{v_{-i}(m), r\} \implies d_i^{m,r}(v) = 0$
- $v_i > \max\{v_{-i}(m), r\} \implies d_i^{m,r}(v) = 1$
- $\tau_i^{m,r}(v) = \begin{cases} med\left\{0, v_{-i}(m) - r, \frac{mr}{n-m}\right\} & \text{if } d_i^{m,r}(v) = 0 \\ med\left\{0, v_{-i}(m) - r, \frac{mr}{n-m}\right\} - \max\{v_{-i}(m), r\} & \text{if } d_i^{m,r}(v) = 1. \end{cases}$

For any non-negative real number r, let $F_{M,r}$ be a family of mechanisms such that for any m, $F_{M,r}^m$ is a maxmed mechanism with reserve price r. Thus, $F_{M,r}$ represents an ex-ante *maxmed sale procedure with reserve price* r. Let $\mathcal{M} := \{F_{M,r}\}_{r \geq 0}$ be the set of all such maxmed sales procedure.

Now, we define a popular strategic axiom in the independent private values setting, strategyproofness, which eliminates any incentive to misreport valuation for each agent by making it weakly dominant strategy to reveal her true valuation in the ensuing message game.

Definition 4. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *strategyproofness* (SP) if for all $i \in N$, all $v_i, v_i' \in \mathbb{R}_+$, and all $v_{-i} \in \mathbb{R}_+^{N \setminus \{i\}}$,

$$u(d_i^m(v_i, v_{-i}), \tau_i^m(v_i, v_{-i}); v_i) \geq u(d_i^m(v_i', v_{-i}), \tau_i^m(v_i', v_{-i}); v_i).$$

Next, we define the axiom of 'non-bossiness in decision' which requires (only) the decision rule in a mechanism to be well-behaved in the sense that no agent is able to influence allotment decision of another agent without changing her own allotment decision.

Definition 5. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *non-bossiness in decision* (NBD) if for all $i \in N$, all $v \in \mathbb{R}_+^N$ and all $v_i' \in \mathbb{R}_+$,

$$d_i^m(v) = d_i^m(v_i', v_{-i}) \implies d_j^m(v) = d_j^m(v_i', v_{-i}), \forall\, j \neq i.$$

and second highest bidder whenever either of their bids is greater than or equal to 20, or else no objects are allocated. Further, any agent who is not allocated an object receives zero transfer, while any agent who is allocated an object pays a price equal to: 20 if bids of all other agents are strictly less than 20, or else the third highest bid. To see that this mechanism is discontinuous, consider a sequence of profiles $\{(20 - \frac{1}{k}, 6, 5)\}_k$. Note that for all k, the agent 2 does not get an object, but she gets an object at the limit profile $(20, 6, 5)$. However, 2 is charged a price 5 at the limit, which makes her prefer getting the object to not getting the object, that is, $u_2((1, -5); 6) > u_2((0, 0); 6)$.

As noted in Thomson [25], NBD represents a strategic hindrance to collusive practices where agents form groups to misreport their valuations in a coordinated manner so that object allotment decision of any one member changes to her benefit, while others' remain unchanged.

The following three axioms represent three different notions of fairness. The first of these states the concept of anonymity which requires that utility derived from an allocation by any agent be independent of her identity.[11] The second one presents the fairness notion that each agent should have some opportunity to win an object, irrespective of other agents' reports.[12] Finally, the third axiom states the notion of no-envy which requires that every agent prefers her own allocation (of decision and transfer from the mechanism) to that of any other agent.[13]

Definition 6. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *anonymity in welfare* (AN) if for all $i \in N$, all $v \in \mathbb{R}_+^N$ and all bijections $\pi : N \mapsto N$,

$$u(d_i(v), \tau_i(v); v_i) = u(d_{\pi i}(\pi v), \tau_{\pi i}(\pi v); \pi v_{\pi i}),$$

where $\pi v := \left(v_{\pi^{-1}(k)}\right)_{k=1}^{n}$.

Definition 7. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *agent sovereignty* (AS) if for all $i \in N$ and all $v \in \mathbb{R}_+^N$, there exists $v_i' \in \mathbb{R}_+$ such that

$$d_i^m(v_i', v_{-i}) = 1.$$

Definition 8. A mechanism $(d^m, \tau^m) \in \mathcal{A}^m$ satisfies *no-envy* (NE) if for all $i \neq j \in N$, all $v \in \mathbb{R}_+^N$,

$$u(d_i(v), \tau_i(v); v_i) \geq u(d_j(v), \tau_j(v); v_i).$$

The next axiom of feasibility requires that the sum of transfers not exceed zero for any profile of valuations and thus, ensures that implementing fair mechanisms do not entail wastage of resources.

Definition 9. A mechanism (d^m, τ^m) satisfies *feasibility* if for all $v \in \mathbb{R}_+^N$,

$$\sum_{i \in N} \tau_i^m(v) \leq 0.$$

[11]This notion has also been used by Ashlagi and Serizawa [2], Athanasiou [3], Hashimoto and Saitoh [13], and Sprumont [23].

[12]Thus agent sovereignty requires that mechanisms be sufficiently sensitive to all agents' valuations so that each agent always has some opportunity to win a unit no matter what other agents' bids are. Moulin [15] states that this axiom is "*reminiscent of the citizen sovereignty of classical social choice.*" As shown in Claim 1 later, it is implied by the other axioms used in our paper.

[13]This notion was introduced by Foley [8] and Varian [26].

Finally, in the axiom below, we present the fairness notion that requires all agents to get a non-negative utility at all possible profiles so that voluntary participation in the mechanism can be ensured.

Definition 10. A mechanism (d^m, τ^m) satisfies *individual rationality* (IR) if for all $i \in N$, all $v \in \mathbb{R}^N_+$,

$$u(d^m_i(v), \tau^m_i(v); v_i) \geq 0.$$

4 Results

We begin by presenting a well known result which states that the decision rule associated with a strategyproof mechanism must be non-decreasing in one's own reported valuation.[14] More specifically, $\forall\, i$ and $\forall\, v_{-i}$, there exists a threshold number $T^m_i(v_{-i})$ such that: i wins an object if $v_i > T^m_i(v_{-i})$, and fails to win an object if $v_i < T^m_i(v_{-i})$.

Fact 1. Any mechanism (d^m, τ^m) satisfies SP and AS, if and only if $\forall\, i \in N$ and $\forall\, v_{-i} \in \mathbb{R}^{N\setminus\{i\}}_+$, there exist functionals $K^m_i : \mathbb{R}^{N\setminus\{i\}}_+ \mapsto \mathbb{R}$ and $T^m_i : \mathbb{R}^{N\setminus\{i\}}_+ \mapsto \mathbb{R}$ such that

$$d^m_i(v) = \begin{cases} 1 & \text{if } v_i > T^m_i(v_{-i}) \\ 0 & \text{if } v_i < T^m_i(v_{-i}) \end{cases} \quad \text{and} \quad \tau^m_i(v) = \begin{cases} K^m_i(v_{-i}) - T^m_i(v_{-i}) & \text{if } d^m_i(v) = 1 \\ K^m_i(v_{-i}) & \text{if } d^m_i(v) = 0 \end{cases}$$

Proof. This result follows from Proposition 9.27 in Nisan [19] and Lemma 1 in Mukherjee [18]. It is also shown as Fact 1 by Basu and Mukherjee [4]. □

Fact 1 allows for arbitrary tie breaking in strategyproof mechanisms for valuation profiles $v \in \mathbb{R}^N_+$ with $v_i = T^m_i(v_{-i})$ for some $i \in N$. In this paper, we use the tie breaking rule below:

> *For any profile v, define $X^m(v) := \{i \in N : v_i > T^m_i(v_{-i})\}$, and $Y^m(v) := \{i \in N : v_i = T^m_i(v_{-i})\}$. At any profile v, if $|Y^m(v)| \leq m - |X^m(v)|$ then all agents in $Y^m(v)$ are allocated an object each, or else the top $m - |X^m(v)|$ agents in $Y^m(v)$ according to the order $1 \succ 2 \succ \ldots \succ n$ are allocated an object each.*[15]

Now, we present a result from Basu and Mukherjee [4]. This result states that any regular family consisting of continuous mechanisms that satisfy AN, AS, NBD and SP; must employ a reserve price r such that top m bidders bidding excess of r win an object when m objects are available for allocation.

[14] This result can be found as Proposition 9.27 in Nisan [19] and Lemma 1 in Mukherjee [18].

[15] Note that irrespective of the tie breaking rule used to decide allocation decisions for any agent i; she is indifferent between getting and not getting the object (as her utility is $K^m_i(v_{-i})$ in both cases). All results in this paper would continue to hold with small notational modifications if a different simple order (other that $1 \succ \ldots \succ n$) is used to break ties in the aforesaid manner.

Fact 2. Fix any family $F \in \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_r$. For any m, if the mechanism F^m satisfies AN, AS, NBD and SP, then there exists an $r \geq 0$ such that for all $i \in N$ and all $v \in \mathbb{R}^N_+$,

- $T_i^m(v_{-i}) = \max\{v_{-i}(m), r\}$ and
- $K_i^m(v_{-i}) = K^m(z)$ where $K^m : \mathbb{R}^{n-1}_+ \mapsto \mathbb{R}$ is a symmetric functional.

Proof. The result follows of from Theorem 3 and Propositions 1 and 2 in Basu and Mukherjee [4]. □

We now present the first result of this paper which investigates families of mechanisms that satisfy AN, AS, NBD and NE.

Proposition 1. Fix any family $F \in \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_r$. For any m, if the mechanism F^m satisfies AN, AS, NBD, NE and SP, then there exists an $r \geq 0$ such that for all $i \in N$ and all $v_{-i} \in \mathbb{R}^{n-1}_+$,

(A) If $v_{-i}(m) \geq r$, then $K^m(v_{-i}) = f(v_{-i}(m))$ where $f() : [r, \infty) \mapsto \mathbb{R}$ is a continuous non-decreasing functional such that for all $x, x' \geq r$,

$$\frac{f(x) - f(x')}{x - x'} \leq 1.[16]$$

(B) If $v_{-i}(m) < r$, then $K^m(v_{-i}) = C^m$ where C^m is an arbitrary real constant.

Proof. Fix any family $F \in \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_r$, any $m \in \mathbb{N}$, and consider the mechanism $F^m := (d^m, \tau^m)$. Suppose that F^m satisfies AN, AS, NBD, NE and SP. We prove each of the statements below.

`Proof of (A)`: From Fact 2 it follows that $\exists\, r \geq 0$ such that $\forall\, z \in \mathbb{R}^{n-1}_+$, $T^m(z) = \max\{z(m), r\}$. Since both $K^m(.)$ and $T^m(.)$ functions are symmetric (as reported in Fact 2), our proof can be accomplished by proving statements (A) and (B) to be true for all $z \in \{z' \in \mathbb{R}^{n-1}_+ : z'(t) = z'_t, \forall\, t = 1, \ldots, n-1\}$. So, fix an arbitrary $z \in \mathbb{R}^{n-1}_+$ such that $z_1 \geq z_2 \geq \ldots \geq z_{n-1}$, and $z_m \geq r$. Observe that to establish $K^m(z) = f(z_m)$ (that is, the image of $K^m(.)$ at point z depends only on the mth coordinate), we need to show that: **(i)** for all $k = 1, \ldots, m-1$, $K^m(x_k, z_{-k}) = K^m(z), \forall\, x_k \geq z_m$ and **(ii)** for all $k = m+1, \ldots, n-1$, $K^m(x_k, z_{-k}) = K^m(z), \forall\, x_k < z_m$.

Suppose that **(i)** does not hold. That is, there exists a $k \in \{1, \ldots, m-1\}$ and an $x'_k > z_m$ such that $K^m(x'_k, z_{-k}) \neq K^m(z)$. Consider the profile v such that $v_1 = x'_k$ and $v_{-1} = z$. Therefore, for all $t = 1, \ldots, n-1$, $v_{t+1} = z_t$ and so, by Fact 2, $d_1(v) = d_{k+1}(v) = 1$. Now, it is well known that in a quasilinear setting, NE implies that any two agents receiving the

[16] That is, f is Lipschitz of degree 1.

same decision also get the same transfer, and so, $\tau_1(v) = \tau_{k+1}(v)$.[17] But this implies that $-z_m + K^m(z) = -z_m + K^m(x'_k, z_{-k})$ and hence, we get a contradiction. Now, suppose that **(ii)** does not hold. That is, there exists a $k \in \{m+1, \dots, n-1\}$ and an $x''_k < z_m$ such that $K^m(x''_k, z_{-k}) \neq K^m(z)$. As before, construct a profile w such that $w_1 = x''_k$ and $w_{-1} = z$. Then, by Fact 2, the top m bidders $\{2, 3, \dots, m+1\}$ win an object each, and so, $d_1(w) = d_{k+1}(w) = 0$, which by NE, implies that $\tau_1(w) = \tau_{k+1}(w) \implies K^m(z) = K^m(x''_k, z_{-k})$. Hence, we again get a contradiction.

Thus, we can infer that for any $v \in \mathbb{R}^N_+$, $K^m(v) = f(v_{-i}(m))$. Now, consider any profile $\bar{v}$ such that $\bar{v}_k = \bar{v}(k)$ for all $k = 1, \dots, n$ and $\bar{v}_{m+1} \geq r$. By Fact 2, $d_m(\bar{v}) = 1, d_{m+1}(\bar{v}) = 0$. Applying the notion of NE for the pair of agents m and $m+1$, we get that $0 \leq K^m(\bar{v}_{-(m+1)}) - K^m(\bar{v}_{-m}) \leq \bar{v}_m - \bar{v}_{m+1}$. From the discussion above, it follows that $0 \leq f(\bar{v}_m) - f(\bar{v}_{m+1}) \leq \bar{v}_m - \bar{v}_{m+1}$. Since, the profile $\bar{v}$ was arbitrarily chosen, we have established that for all $x \geq y \geq r$, $f(x) \geq f(y)$. That is $f(.)$ is a non-decreasing function with a slope less than 1 over the interval $[r, \infty)$. Further, we can see that $f(.)$ is Lipschitz of degree 1, and so, it follows that $f(.)$ is continuous.

`Proof of (B):` As before, note that Fact 2 implies that $\exists\, r \geq 0$ such that $\forall\, z \in \mathbb{R}^{n-1}_+$, $T^m(z) = \max\{z(m), r\}$. Define for any $z \in \mathbb{R}^{n-1}_+$, $k^z := |\{z_i : z_i \geq r\}|$ to be the number of coordinates of z weakly greater than r. Also define for all $t = 0, \dots, n-1$, $S^t := \{z \in \mathbb{R}^{n-1}_+ : k^z = t\}$ to the set of vectors in $\mathbb{R}^{n-1}_+$ which have exactly t coordinates that are weakly greater than r. Therefore, the set of all $z \in \mathbb{R}^{n-1}_+$ such that $z(m) < r$ is $S^* := \cup_{t=0}^{t=m-1} S^t$. In the following three steps, we show that for all $z \in S^*$, $K^m(z) = C^m$ where C^m is an arbitrary real constant.

STEP 1: For all $z \in S^0$, $K^m(z) = C^m$ where C^m is an arbitrary real constant.

Proof of Step: Suppose there exists $z, z' \in S^0$ such that $K^m(z) \neq K^m(z')$. Construct a sequence of profiles $\{{}^t v\}_{t=1}^{n-1}$ such that ${}^1 v_1 = z'_1$, ${}^1 v_{-1} = z$, and for all $2 \leq t \leq n-1$, ${}^t v_t = z'_t$ with ${}^t v_{-t} ={}^{t-1} v_{-t}$. Since for all t, ${}^t v \in [0, r)^n$, by Fact 2, $d_i({}^t v) = 0$ for all $i \in N$. Since no agent wins an object at any member of sequence $\{{}^t v\}$, by NE, we get that for all $i \neq j \in N$, $\tau_i({}^t v) = \tau_j({}^t v) \implies K^m({}^t v_{-i}) = K^m({}^t v_{-j})$ for all t. Therefore, $K^m({}^1 v_{-1}) = K^m({}^2 v_{-2}) = \dots = K^m({}^{n-1} v_{-\{n-1\}})$. Since, by definition, ${}^1 v_{-1} = z$ and ${}^{n-1} v_{-\{n-1\}} = z'$, we get $K^m(z) = K^m(z')$, which contradicts our supposition.

STEP 2: For any $t = 1, \dots, m-1$ and any $z \in S^t$, $K^m(z) = C^{mt}$ where C^{mt} is an arbitrary real constant.

Proof of Step: Fix a $t = 1 \dots, m-1$ and consider any $z \in S^t$. As before, without loss of generality, suppose that $z_k = z(k), \forall\, k = 1, \dots, n-1$. We first prove that (i) for all $k = 1, \dots, t$, all $x_k > z_k$, $K^m(x_k, z_{-k}) = K^m(z)$; and (ii) for all $k = t+1, \dots, n-1$, all $x_k < z_k$, $K^m(x_k, z_{-k}) = K^m(z)$. Suppose that (i) is not true. Then there exists a

[17] Note that for any profile v and any $i \neq j \in N$, NE implies that $v_i(d_i(v) - d_j(v)) \geq \tau_j(v) - \tau_i(v) \geq v_j(d_i(v) - d_j(v))$, and so, $d_i(v) = d_j(v) \implies \tau_i(v) = \tau_j(v)$.

$k \in \{1, \ldots, t\}$ and an $x'_k > z_k$ such that $K^m(x'_k, z_{-k}) \neq K^m(z)$. Consider the profile v such that $v_1 = x'_k$, $v_2 = z_k$ and $v_{-1-2} = z_{-k}$. Note that by construction, $z_k \geq r$. If $z_k > r$ then by Fact 2, $d_1(v) = d_2(v) = 1$, and so, from NE it follows that $\tau_1(v) = \tau_2(v)$, which implies $K^m(z) = K^m(x_k, z_{-k})$ that contradicts our supposition. On the other hand, if $z_k = r$, then by Fact 2 and our tie-breaking rule, $d_1(v) = d_2(v) = 1$ and so, as before, from NE we get that $K^m(x_k, z_{-k}) = K^m(z)$. Similarly, suppose that (ii) is not true and so, there exists $l \in \{t+1, \ldots, n-1\}$ and an $x'_l < z_l$ such that $K^m(x'_l, z_{-l}) \neq K^m(z)$. As before, construct a profile v' such that $v'_1 = x'_l$, $v'_2 = z_l$ and $v'_{-1-2} = z_{-l}$. By Fact 2, $d_i(v') = d_2(v') = 0$, and so, by NE, it follows that $\tau_1(v') = \tau_2(v') \implies K^m(z) = K^m(x'_l, z_{-l})$, which is a contradiction. Hence, the result follows.

STEP 3: For all $t = 1, \ldots, m-1$, $C^{mt} = C^m$.
Proof of Step: We accomplish this proof by induction. We first show that $C^{m1} = C^m$. To see this, consider any profile v such that $v_1 \geq r > v_2 \geq \ldots \geq v_n$. Therefore, $v_{-1} \in S^0$ and for all $j \neq 1$, $v_{-j} \in S^1$. Therefore, by the earlier steps, $K^m(v_{-1}) = C^m$ and $K^m(v_{-j}) = C^{m1}$ for all $j \neq 1$. Further, by Fact 2, $d_1(v) = 1, d_2(v) = 0$. Therefore, NE implies that $v_2 - r \leq C^{m1} - C^m \leq v_1 - r$. Since v_1 and v_2 could have been chosen arbitrarily close to r satisfying $v_1 > r > v_2$, it must be that $C^{m1} = C^m$. Now, suppose that $C^{mt} = C^m$ for some t such that $0 \leq t \leq m-2$. We shall then show that $C^{mt+1} = C^m$. To see this, consider the profile v such that $v_1 \geq \ldots \geq v_{t+1} > r > v_{t+2} \geq \ldots \geq v_n$. By the earlier steps, $v_{-\{t+1\}} \in S^t \implies K^m(v_{-\{t+1\}}) = C^{mt} = C^m$ and $v_{-\{t+2\}} \in S^{t+1} \implies K^m(v_{-\{t+2\}}) = C^{mt+1}$. Also, by Fact 2, $d_{t+1}(v) = 1, d_{t+2}(v) = 0$. Then, by NE, $v_{t+2} - r \leq C^{mt+1} - C^m \leq v_{t+1} - r$. Since, v_{t+1} and v_{t+2} can be chosen arbitrarily close to r satisfying $v_{t+2} < r < v_{t+1}$, it must be that $C^{mt} = C^m$. □

Now, we present the second result of this paper which investigates strategyproof and feasible mechanisms that satisfy AN, AS, NBD and IR.

Proposition 2. Fix any family $F \in \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_r$. For any $m \in \mathbb{N}$, if the mechanism F^m satisfies AN, AS, feasibility, IR, NBD, NE and SP, then then there exists $r \geq 0$ such that for all $i \in N$ and all $v_{-i} \in \mathbb{R}^{n-1}_+$,

(A) When $r > 0$; $K^m(v_{-i}) = 0$ if $v_{-i}(m) < r$, or else $K^m(v_{-i}) = f(v_{-i}(m))$ where where $f(.) : [r, \infty) \mapsto \left[0, \frac{mr}{n-m}\right]$ is a non-decreasing 1-Lipschitz continuous function.

(B) When $r = 0$; $K^m(v_{-i}) = 0$.

(C) For any $r \geq 0$, $K^m(.)$ is continuous.

Proof: Fix any family $F \in \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_r$, any m, and consider the mechanism $F^m := (d^m, \tau^m)$. Suppose that F^m satisfies AN, AS, feasibility, IR, NBD, NE and SP. As noted earlier, Fact 2 implies that $\exists\, r \geq 0$ such that $\forall\, z \in \mathbb{R}^{n-1}_+$, $T^m(z) = \max\{z(m), r\}$. Now, we prove each of the statements below.

`Proof of (A)`: Suppose $r > 0$. Given Proposition 1, to establish this statement, we need to show that IR and feasibility imply the following two restrictions on the mechanisms characterized by Proposition 1: (i) $C^m = 0$ and (ii) $f(x) \in \left[0, \frac{mr}{n-m}\right]$ for all $x \geq r > 0$. To establish (i) consider a profile $v \in [0, r)^n$. Fact 2 implies that $d_i^m(v) = 0, \forall\, i$ and so, by Proposition 1, $u(d_i^m(v), \tau_i^m(v); v_i) = K^m(v_{-i}) = C^m$ for all $i \in N$. Therefore, IR implies that $C^m \geq 0$, while feasibility implies that $nC^m \leq 0$. Thus, we get that $C^m = 0$. To prove (ii), fix any $x \geq r$ and consider the profile v^x such that $v_1^x \geq \ldots \geq v_m^x = x \geq r > v_{m+1}^x \geq \ldots \geq v_n^x$. Proposition 1 and the property (i) proved above, imply that for all $i \in \{1, \ldots, m\}$, $K^m(v_{-i}^x) = 0$ and for all $i \in \{m+1, \ldots, n\}$, $K^m(v_{-i}^x) = f(x)$. Further, by Fact 2 and IR, for all $i \in \{m+1, \ldots, n\}$, $u(d_i^m(v^x), \tau_i^m(v^x); v_i^x) = K^m(v_{-i}^x) = f(x) \geq 0$. On the other hand, feasibility implies that $(n-m)f(x) + m(0-r) \leq 0 \implies f(x) \leq \frac{mr}{n-m}$. Since $x \geq r$ was chosen arbitrarily, by Proposition 1, the result follows. □

`Proof of (B)`: Suppose $r = 0$. Then by Proposition 1, for all $v \in \mathbb{R}^N_+$ and all $i \in N$, $K^m(v_{-i}) = f(v_{-i}(m))$. Suppose there exists $y > 0$ such that $f(y) > 0$. Consider profile v where $v_k = v(k)$ for all $k = 1 \ldots, n$, $v_m = y$ and $v_{m+1} = \epsilon \in \left(0, \min\left\{y, \frac{(n-m)f(y)}{m}\right\}\right)$. Therefore, by Fact 2, $d_i^m(v) = 1$ for all $i = 1, \ldots, m$ and $d_i^m(v) = 0$ for all $i = m+1, \ldots, n$. Further, by Proposition 1 and Fact 2, $\tau_i^m(v) = K^m(v_{-i}) - T^m(v_{-i}) = f(\epsilon) - \epsilon$ for all $i \in \{1, \ldots, m\}$, and $\tau_i^m(v) = K^m(v_{-i}) = f(y)$ for all $i \in \{m+1, \ldots, n\}$. Now, $r = 0$ implies that $u_{m+1}(d_{m+1}^m(\bar{\epsilon}^n), \tau_{m+1}^m(\bar{\epsilon}^n); \epsilon) = f(\epsilon)$, and so, by IR, $f(\epsilon) \geq 0$. But then, by the particular construction of ϵ, the sum of transfers at profile v becomes positive, which contradicts feasibility. Further, it follows trivially from IR and feasibility that $f(0) = 0$ when $r = 0$.[18] Therefore, we get that whenever $r = 0$, $f(y) = 0, \forall\, y \geq 0$. Thus, the result follows. □

Proof of (C): Note that this result follows trivially when $r = 0$. When $r > 0$, given Proposition 1, to prove this statement we need to establish that $f(r) = 0$. Suppose not; that is, suppose that $f(r) > 0$. By continuity, there exists a $\gamma > 0$ such that for all $\delta \in (0, \gamma)$, $f(r + \delta) > 0$. Further, there exists a $\delta' \in (0, \gamma)$ such that $0 < \delta' < f(r + \delta')$ (or else for all $\delta \in (0, \gamma)$, $0 < f(r + \delta) \leq \delta$ implying that $f(r) = 0$ which would be a contradiction). Now, consider a profile v such that for all $k = 1, \ldots, n$, $v_k = v(k)$, $v_m = r + \delta'$ and $v_{m+1} < r$. Note that by statement (A), $K^m(v_{-m}) = 0$ since $v_{-m}(m) = v_{m+1} < r$. Further, by Fact 2, $d_m(v) = 1$ and $d_{m+1}(v) = 0$, and so, $u(d_m(v), \tau_m(v); v_m) = \delta' < f(r + \delta') = K^m(v_{-\{m+1\}}) = u(d_{m+1}(v), \tau_{m+1}(v); v_{m+1})$, which contradicts NE. Hence, the result follows. □

Now, we proceed to the main result of our paper, which is the characterization of

[18] By considering the profile $\bar{0}^n$.

the multiple object version of the maxmed mechanisms introduced by Sprumont [23] for a single object setting. We first define the notion of Pareto dominance in a class of mechanisms. For any given supply of objects m, and any set of mechanisms S^m, define a *weak partial order* $\succeq$ on S^m in the following manner. For any two mechanisms $(d^m, \tau^m), (d'^m, \tau'^m) \in S^m$, let $(d^m, \tau^m) \succeq (d'^m, \tau'^m)$ iff for all $i \in N$ and all $v \in \mathbb{R}^N_+$, $u(d^m_i(v), \tau^m_i(v); v_i) \geq u(d'^m_i(v), \tau'^m_i(v); v_i)$. If in addition, this inequality is strict for some i and some v, then we write that $(d^m, \tau^m) \succ (d'^m, \tau'^m)$ and say that (d^m, τ^m) *Pareto dominates* (d'^m, τ'^m). On the other hand, if $u(d^m_i(v), \tau^m_i(v); v_i) = u(d'^m_i(v), \tau'^m_i(v); v_i)$ for all i and all v, then we write that $(d^m, \tau^m) \sim (d'^m, \tau'^m)$ and say that (d^m, τ^m) is *Pareto equivalent* to (d'^m, τ'^m). Finally, we call the class of mechanisms in S^m that are not dominated by any other mechanism in S^m, as the set of *Pareto optimal* mechanisms in S^m.

Now, we define our notion of Pareto optimal families of mechanisms. For any given set of families $\mathcal{F}$, define a weak partial order '$\mathbf{r}$' over $\mathcal{F}$, where for any $F, G \in \mathcal{F}$, $F \,\mathbf{r}\, G$ iff $F^m \succeq G^m$ for all m. If in addition, there exists an m' such the $F^{m'} \succ G^{m'}$, then we write that $F \,\mathbf{s}\, G$ and say that F Pareto dominates G. Also, if $F^m \sim G^m$ for all m, we write $F \,\mathbf{e}\, G$ and say that F is Pareto equivalent to G. Thus, we call the set of families in $\mathcal{F}$ that are Pareto undominated by any other family in $\mathcal{F}$, as the set of Pareto optimal families in $\mathcal{F}$.

Now, consider the set of families $\tilde{\mathcal{A}} \subseteq \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_r$ such that for any $F \in \tilde{\mathcal{A}}$ and any m, the mechanism F^m satisfies AN, feasibility, IR, NBD, NE and SP. Let $\mathcal{A}^*$ be the set of Pareto optimal families in $\tilde{\mathcal{A}}$. We first show below that all families in $\mathcal{A}^*$ must comprise of mechanisms that satisfy AS, and then use this result to establish that $\mathcal{A}^*$ is same as the set of maxmed sales procedures $\mathcal{M}$.[19]

Claim 1. *For any family $F \in \mathcal{A}^*$, and any m, the mechanism $F^m := (d^m, \tau^m)$ satisfies AS.*

Proof: Fix any $F \in \mathcal{A}^*$, and suppose that there exists an $\hat{m}$ such that $F^{\hat{m}} = (d^{\hat{m}}, \tau^{\hat{m}})$ does not satisfy AS. Let $K^{\hat{m}}(.)$ and $T^{\hat{m}}(.)$ be the functions associated with $F^{\hat{m}}$ as described in Fact 1. By our supposition, there must exist a $z \in \mathbb{R}^{n-1}_+$ such that either $T^{\hat{m}}(z) = \infty$ or else $T^{\hat{m}}(z) < 0$. Now if $T^{\hat{m}}(z) < 0$, then at any profile v where $v_1 = 0$ and $v_{-1} = z$, by Fact 1, $d^{\hat{m}}_1(v) = 1$, and so, by NE (that is, the inequalities presented in footnote 17), $d^{\hat{m}}_i(v) = 1$ for all $i \neq 1$, which is a contradiction since supply of objects is always less than total demand (see footnote 8).
So, the only remaining possibility is that $T^{\hat{m}}(z) = \infty$. Now, fix any i, and any profiles v, v' such that $v'_{-i} = v_{-i} = z$, and $v'_i > \max_{t=1,\ldots,n-1} z_t$. By Fact 1, $d^{\hat{m}}_i(v) = d^{\hat{m}}_i(v') = 0$, and so, by NBD, $d^{\hat{m}}_j(v) = d^{\hat{m}}_j(v')$ for all $j \neq i$. As argued above, by NE, $d^{\hat{m}}_j(v') = 0$ implies

[19] See Definition 3.

that for all $j \neq i$, $d_j^{\hat{m}}(v') = 0$, which then implies that $d_j^{\hat{m}}(v) = 0, \forall\, j \neq i$. Since, i and v_i were chosen arbitrarily, we get that for any v,

$$\exists\, i \ni v_{-i} = z \implies d_j^{\hat{m}}(v) = 0, \forall\, j.$$

Further, since $F \in \bar{\mathcal{A}}_r$, we can infer that for any v,

$$\exists\, i \ni v_{-i} = z \implies d_j^{m}(v) = 0, \forall\, j, \forall\, m.$$

Now construct a family of mechanisms $\bar{F}$ such that for any m, the mechanism $\bar{F}^m := (\bar{d}^m, \bar{\tau}^m)$ is defined such that $(d^m(v), \tau^m(v)) = (\bar{d}^m(v), \bar{\tau}^m(v))$ for all v satisfying the property $v_{-j} \neq z, \forall\, j$; or else,

- for any i, $\bar{d}_i^m(v) = \begin{cases} 1 & \text{if } v_i > z(m) \\ 0 & \text{if } v_i < z(m), \end{cases}$
- for any i, $\bar{\tau}_i^m(v) = \begin{cases} K^m(v) + \frac{z(m)}{n} - z(m) & \text{if } \bar{d}^m(v) = 1 \\ K^m(v) + \frac{z(m)}{n} & \text{if } \bar{d}^m(v) = 0. \end{cases}$

It is easy to see that for any m, $\bar{F}^m \succ F^m$ since for all v,

$$\begin{aligned} \nexists\, j \ni v_{-j} \neq z &\implies u_i(d^m(v), \tau^m(v); v_i) = u_i(\bar{d}^m(v), \bar{\tau}^m(v); v_i), \forall\, i, \\ \exists\, j \ni v_{-j} \neq z &\implies u_i(d^m(v), \tau^m(v); v_i) < u_i(\bar{d}^m(v), \bar{\tau}^m(v); v_i), \forall\, i. \end{aligned}$$

Therefore, by construction, $\bar{F}$ **s** F. Now, it is easy to see that $\bar{F} \in \bar{\mathcal{A}}_r \cap \bar{\mathcal{A}}_c$. Further, we can easily check that for any m, the mechanism $\bar{F}^m$ satisfies feasibility, IR, NE, NBD and SP, since F^m too satisfies all these properties. Thus, $\bar{F} \in \tilde{\mathcal{A}}$, which contradicts $F \in \mathcal{A}^*$. □

Thus, as shown in Claim 1 above, we get AS for free from the set of other axioms. This brings us to main result of the paper below, which shows that maxmed sales procedures are the only families that are Pareto optimal in $\tilde{\mathcal{A}}$.

Theorem 1. $\mathcal{M} = \mathcal{A}^*$.

Proof. We prove the necessity and the sufficiency components of this result separately below.

`Necessity.` Fix any family $F \in \mathcal{A}^*$, any m and consider the mechanism $F^m := (d^m, \tau^m)$. Suppose that F^m satisfies AN, feasibility, IR, NBD, NE and SP. Let $K^m(.)$ and $T^m(.)$ be the functions associated with F^m as described by Fact 2. From Claim 1, Facts 1 and 2, it follows that $\exists\, r \geq 0$ such that $\forall\, m$ and $\forall\, z \in \mathbb{R}_+^{n-1}$, $T^m(z) = \max\{z(m), r\}$. Further, Propositions 1 and 2 imply that **(i)** $v_{-i}(m) \leq r \implies K^m(v_{-i}) = 0, \forall i, \forall v$. Now suppose that there exists $z \in \mathbb{R}_+^{n-1}$ with $z(m) \in \left(r, \frac{nr}{n-m}\right)$ and $K^m(z) < z(m) - r$. Then, define the set $P_z := \{v \in \mathbb{R}^N_+ | \exists\ i \in N$ such that $v_{-i} = z\}$, and for all $v \in P_z$, define the set $a^v_z := \{i \in N | v_{-i} = z\}$. Therefore, by Fact 1, P_z is the set of all possible profiles v such that all agents i in a^v_z are assigned the following transfer by mechanism F^m,

$$\tau^m_i(v) = \begin{cases} K^m(z) & \text{if } d^m_i(v) = 0 \\ K^m(z) - \max\{z(m), r\} & \text{otherwise.} \end{cases}$$

Now, construct another family F'' such that $F''^k = F^k$ for all $k \neq m$, and $F''^m := (d''^m, \tau''^m)$ satisfies the following properties:

- $(d''^m{}_i(v), \tau''^m{}_i(v)) = (d^m_i(v), \tau^m_i(v))$ for all $i \in N$ and all $v \in \mathbb{R}^N_+ \setminus P_z$,
- $d''^m m_i(v) = dm_i(v)$ for all $i \in N$ and all $v \in P_z$,
- $\tau''^m_i(v) = \tau^m_i(v)$ for all $i \notin a^v_z$, and all $v \in P_z$,
- for all $v \in P_z$ and all $i \in a^v_z$,

$$\tau''^m{}_i(v) = \begin{cases} z(m) - r & \text{if } d''^m{}_i(v) = 0 \\ z(m) - r - \max\{z(m), r\} & \text{otherwise.} \end{cases}$$

Since by supposition, $K^m(z) < z(m) - r$, it can easily be seen that (d''^m, τ''^m) Pareto dominates (d^m, τ^m), and so, we can infer that F'' **s** F. Further, it is easy to check that $F \in \tilde{\mathcal{A}} \implies F'' \in \tilde{\mathcal{A}}$. Thus, we get a contradiction to $F \in \mathcal{A}^*$. Therefore, $K^m(z) = z(m) - r$ for all $z \in \mathbb{R}^{n-1}_+$ with $z(m) \in \left(r, \frac{nr}{n-m}\right)$.
Similarly, by Proposition 2, we can argue that for all $z \in \mathbb{R}^{n-1}_+$ with $z(m) \geq \frac{nr}{n-m}$, $0 \leq K^m(z) \leq \frac{mr}{n-m}$. If there exists a z' such that $K^m(z') < \frac{mr}{n-m}$, then as argued above, we can show that F is Pareto dominated by another suitably constructed family in $\tilde{\mathcal{A}}$, which would be a contradiction to $F \in \mathcal{A}^*$. Hence, for any $z \in \mathbb{R}^{n-1}_+$, $K^m(z) = \frac{mr}{n-m}$ whenever $z(m) > \frac{nr}{n-m}$, or equivalently, $z(m) - r > \frac{mr}{n-m}$. By the continuity of $K^m(.)$ function proved in Proposition 2, it now follows that for any $z \in \mathbb{R}^n_+$, $K^m(z) = med\{0, z(m) - r, \frac{mr}{n-m}\}$, and so, $F \in \mathcal{M}$.

`Sufficiency:` Fix any $r \geq 0$, any family $F_{M,r}$, any $m \in \mathbb{N}$, and consider the maxmed mechanism $F^m_{M,r} := (d^{m,r}, \tau^{m,r})$. Note that, for any profile v such that $v_{-i}(m) \leq r$; for any $i \in N$, $\tau^{m,r}_i(v) = 0$ if $d^{m,r}_i(v) = 0$ or else $\tau^{m,r}_i(v) = -r$. Similarly, for any profile v such that $v_{-i}(m) \in \left(r, \frac{nr}{n-m}\right)$; for any $i \in N$, $\tau^{m,r}_i(v) = v_{-i}(m) - r$ if $d^{m,r}_i(v) = 0$ or else $\tau^{m,r}_i(v) = -r$. Finally, for any profile v such that $v_{-i}(m) \geq \frac{nr}{n-m}$; for all $i \in N$, $\tau^{m,r}_i(v) = \frac{mr}{n-m}$ if $d^{m,r}_i(v) = 0$ or else $\tau^{m,r}_i(v) = \frac{mr}{n-m} - v_{-i}(m)$. Thus, it can easily be seen that for a given threshold function $T^m(v) = \max\{v_{-i}(m), r\}, \forall v$, all agents with the same allotment decision receive the same transfer at all profiles. Also, for any agents $i \neq j$, and any v with $d_i(v) = 1, d_j(v) = 0$, we get that $v_i \geq \tau^{m,r}_j(v) - \tau^{m,r}_i(v) \geq v_j$ in all the above cases. Thus, we can infer that $(d^{m,r}, \tau^{m,r})$ satisfies feasibility, NE and IR. Further, it is easy to see that $(d^{m,r}, \tau^{m,r})$ satisfies NBD, and SP. To see that $(d^{m,r}, \tau^{m,r})$ satisfies continuity, note that the premise of the continuity condition applies only if the limit profile $\tilde{v}$ (of the chosen sequence) is such that there exists an i such that $\tilde{v}_i = \max\{\tilde{v}_{-i}(m), r\}$, in which case $u((1, \tau_i(\tilde{v}; d_i = 1); \tilde{v}_i) = u((0, \tau_i(\tilde{v}; d_i = 0); \tilde{v}_i) = med\left\{0, v_{-i}(m) - r, \frac{mr}{n-m}\right\}$. Finally, it is easy to see that $F_{M,r} \in \bar{\mathcal{A}}_r$, because the set of profiles where no object gets allocated is $[0, r)^n$, which remains unchanged as m increases.

To complete the proof of sufficiency, we now need to show that $F_{M,r}$ is Pareto undominated in $\tilde{\mathcal{A}}$. To prove this, suppose the contrapositive, that is, suppose that there exists a family of mechanisms $\hat{F} \in \tilde{\mathcal{A}}$ such that $\hat{F}$ **s** $F_{M,r}$. This supposition implies that there exists an m' such that $\hat{F}^{m'} := (\hat{d}^{m'}, \hat{\tau}^{m'}) \succ F^{m'}_{M,r} = (d^{m',r}, \tau^{m',r})$. Now, since $\hat{F} \in \tilde{\mathcal{A}}$, we can infer from Proposition 2 that there exists an $\hat{r} \geq 0$ such that for all i and all v, the associated threshold function $\hat{T}^{m'}(v_{-i}) = \max\{v_{-i}(m'), \hat{r}\}$ and the associated $\hat{K}^{m'}$ function satisfy the conditions **(A)**, **(B)**, and **(C)** of Proposition 2. Now, from the proof of necessity we can infer that for any k objects, $F^k_{M,\hat{r}}$ is either Pareto equivalent to $\hat{F}^k$ or else Pareto dominates $\hat{F}^k$. Therefore, we can infer that $F_{M,\hat{r}}$ **r** $\hat{F}$, and so, by supposition, $F_{M,\hat{r}}$ **s** $F_{M,r}$. This implies that $\hat{r} \neq r$. If $\hat{r} > r$, then fix $m = 2$ and consider a profile $\bar{v}$ such that $\bar{v}_1 > \ldots > \bar{v}_n$ and for all i, $\bar{v}_i \in \left(r, \min\left\{\frac{nr}{n-m}, \hat{r}\right\}\right)$. It is easy to see that $u(F^2_{M,r}(\bar{v}); \bar{v}_n) = v_m - r > u(F^2_{M,\hat{r}}(\bar{v}); \bar{v}_n) = 0$, which contradicts $F_{M,\hat{r}}$ **s** $F_{M,r}$. Similarly, if $\hat{r} < r$, then again fix $m = 2$ and consider a profile $\tilde{v}$ such that for all i, $\tilde{v}_i = \frac{nr}{n-m} + 1$. Once again we get that $u(F^2_{M,r}(\bar{v}); \bar{v}_n) = \frac{mr}{n-m} > u(F^2_{M,\hat{r}}(\bar{v}); \bar{v}_n) = \frac{m\hat{r}}{n-m}$, which contradicts $F_{M,\hat{r}}$ **s** $F_{M,r}$. Thus, we get a contradiction in both cases, which implies that $F_{M,r}$ is Pareto undominated in $\tilde{\mathcal{A}}$, and so, $F_{M,r} \in \mathcal{A}^*$. □

Now, it is easy to see that no individually rational mechanism can be Pareto dominated by another mechanism that does not satisfy individual rationality. Hence, we can easily infer that within the class of families $\hat{\mathcal{A}} \subset \bar{\mathcal{A}}_c \cap \bar{\mathcal{A}}_c$ that comprise of mechanisms satisfying AN, feasibility, NBD, NE and SP; the set of maxmed families $\mathcal{M}$ is Pareto optimal - but not uniquely Pareto optimal.

5 Conclusion

In this paper, we provide an extension of maxmed mechanisms to the multiple homogeneous objects setting. We conduct our analysis in terms of families of mechanisms which we interpret as ex-ante sale procedures that list a separate mechanism to be used to allocate different possible supplies of the homogeneous objects.

We consider a regular class of families of continuous mechanisms that satisfy anonymity, feasibility, individual rationality, no-envy, non-bossiness in decision and strategyproofness. We show that the maxmed sale procedures, that is, the families which use maxmed mechanisms to allocate any supply of objects while using the same reserve price - are the only Pareto undominated families in the aforesaid class.

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